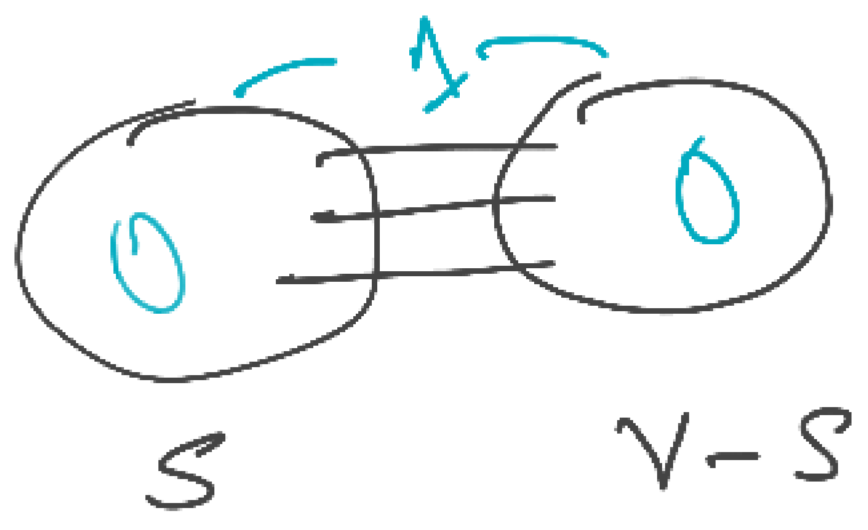
Cuts and Metrics

Metric on V is a function $d(u,v)$:

- $d(u,v) = 0 \iff u = v$
- $d(u,v) = d(v,u)$
- $d(u,v) \leq d(u,w) + d(w,v) \quad \forall w$

"distance function"

Semi-Metric: $d(u,v) = 0$ even if $u \neq v$



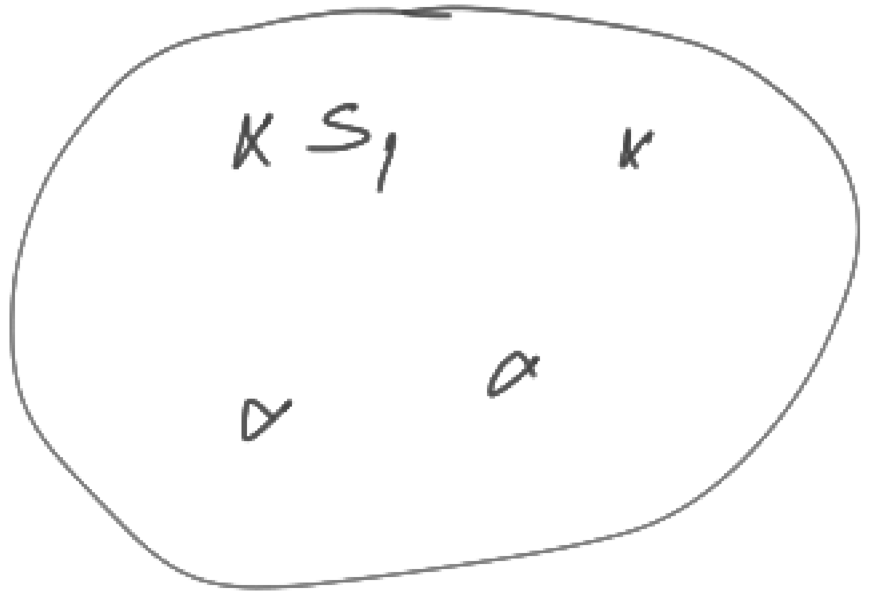
cut-semimetric

Multiway Cut

Given a graph G with edge costs c_e and k terminals $s_1 \cdots s_k$

Delete a minimum cost set of **edges** that separate all terminals.

$\boxed{\text{minimal}}$



$F \to$ some, feasible sol^n, $F \subseteq E$

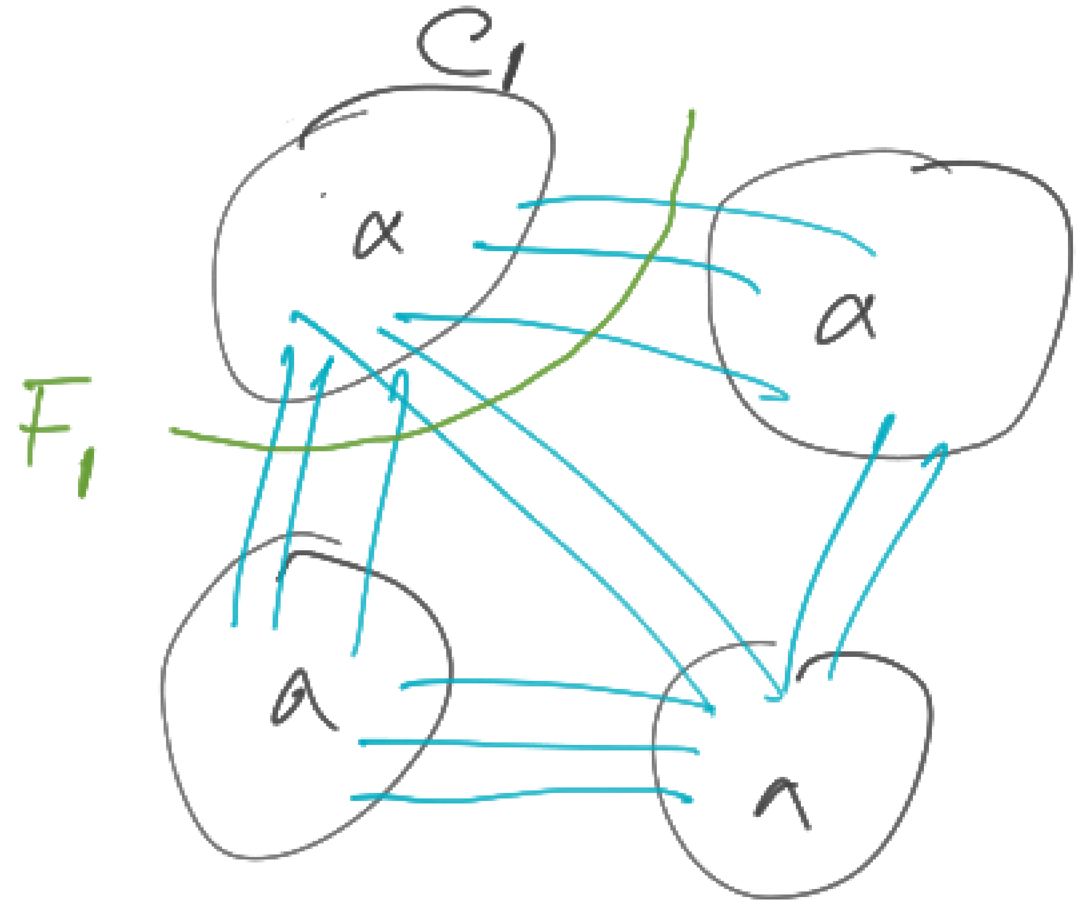
$C_i \to$ set of vertices reachable from s_i in $G-F$

$F_i \to \delta(C_i)$

$\hookrightarrow$ edges with exactly 1 endpoint in C_i

$\downarrow$

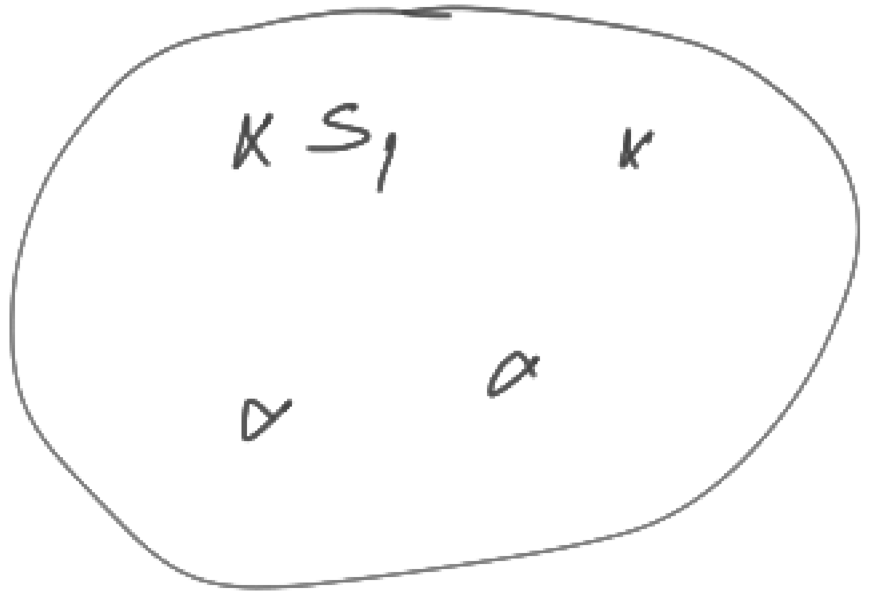
edges of F with an endpt in C_i



Multiway Cut

Given a graph G with edge costs c_e and k terminals $\{s_1 \cdots s_k\} = S$

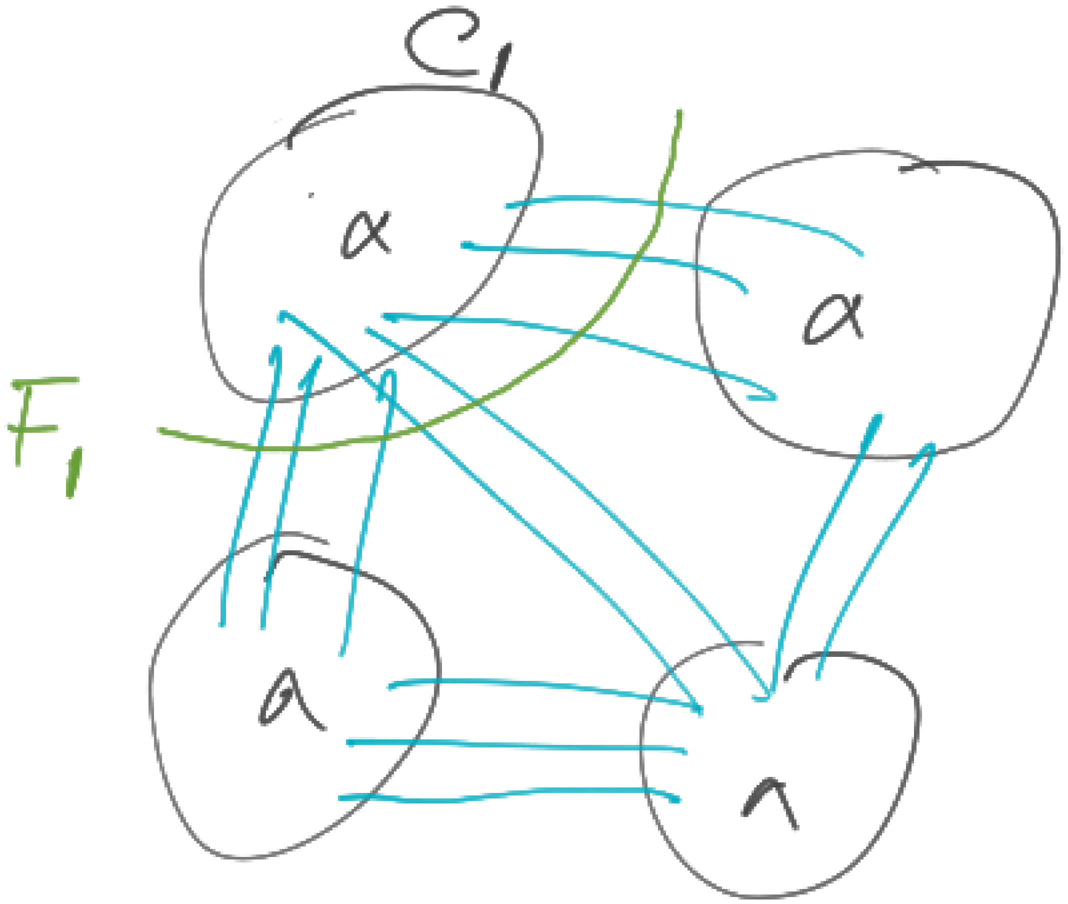
Delete a minimum cost set of edges that separate all terminals.



- $F \subseteq E$ is some feasible sol^n
- C_i = vertices reachable from s_i in $G-F$
- $F_i = \delta(C_i) \longrightarrow$ a cut separating s_i from $S - \{s_i\}$

$\downarrow$

edges CROSSING $(C_i, V - C_i)$



Multiway Cut

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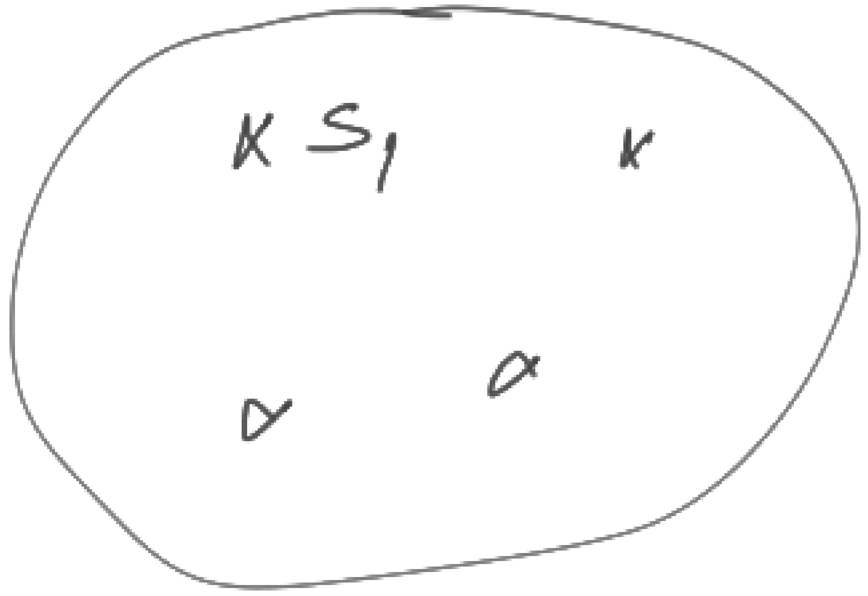
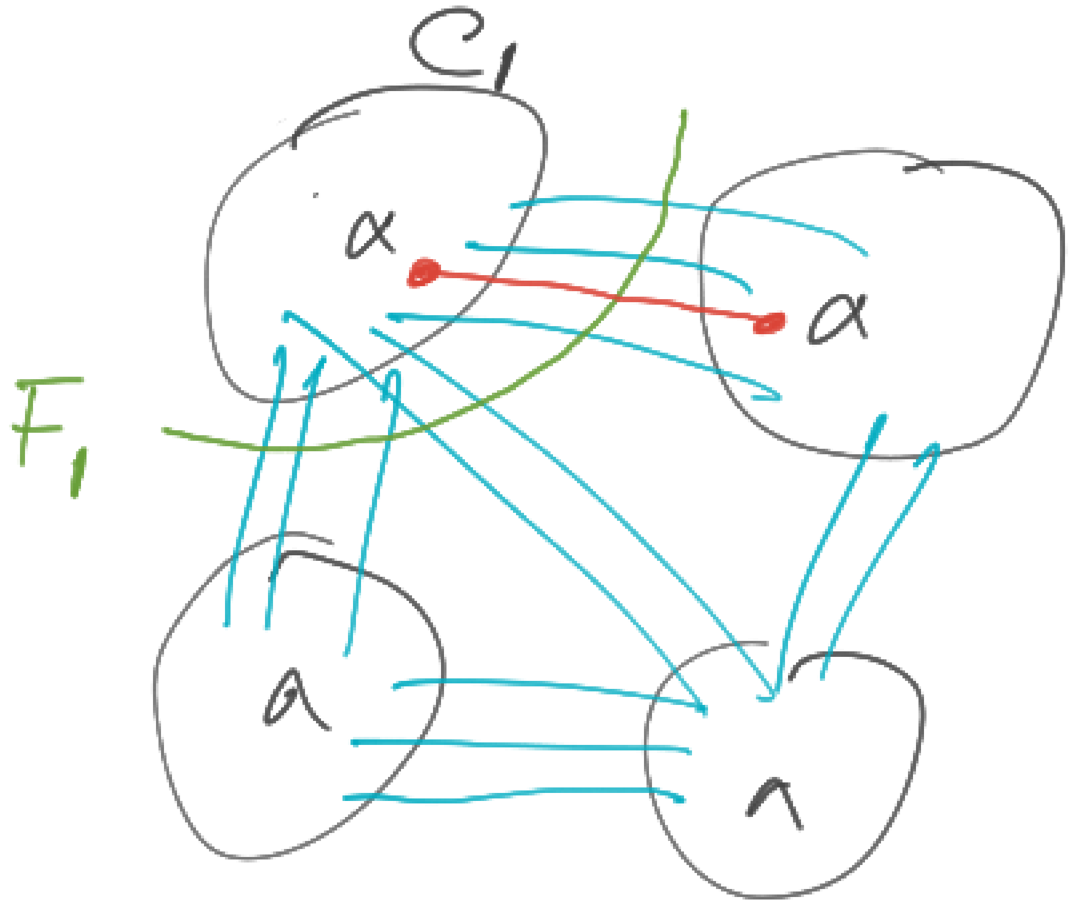


- Algo: $X_i = s_i-(S-\{s_i\})$ mincut

$$X = \bigcup_{i=1}^{k} X_i$$

- Thm: X is a 2-approx



Proof: $F \rightarrow$ optimum sol^n

cardinality

F_i is a $s_i-(S-s_i)$ cut

$\Rightarrow |X_i| \leq |F_i|$

$\therefore |X| \leq \sum |X_i| \leq \sum |F_i| \leq 2|F|$

each $e \in F$ is in exactly 2 F_i.

Edge Costs.

Better approx via LPs

- Find a k-partition of V such that $s_i \in C_i$ and total cost of cross edges is minimized.

- $\forall v \in V$ we have var x_v^i indicating if $v \in C_i$ or not → exactly one of these is 1
- $\forall e \in E$ we have var z_e^i indicating if $e \in \delta(C_i)$ or not → either 0 or 2 of these are 1

- <u>Objective:</u> $\min \frac{1}{2} \sum_e c_e \sum_{i=1}^{k} z_e^i$

such that $\forall v \in V \quad \sum x_v^i = 1$ ← vertex in 1 part

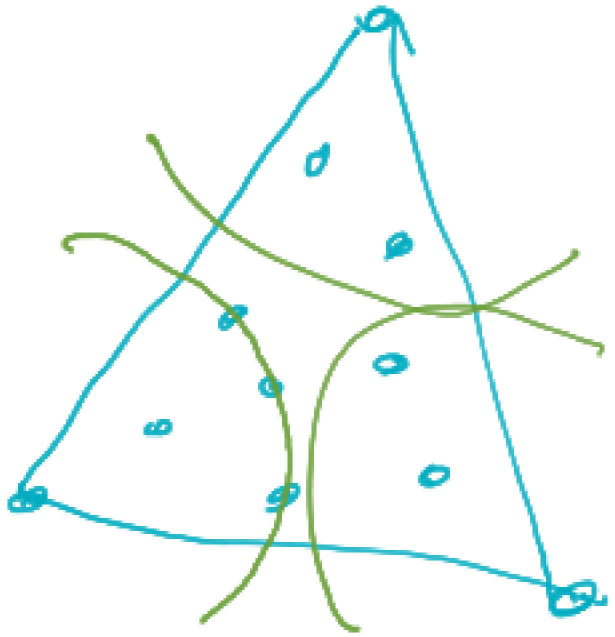
$\forall e = (u,v) \quad z_e^i \geq |x_u^i - x_v^i|$ ← cross edges must be paid for

$\forall s_i \quad x_{s_i}^i = 1$ ← terminals in separate parts.

More compact form using metrics

- Given two vectors $x, y \in \mathbb{R}^k$ $\quad \|x-y\|_1 = \sum_{i=1}^{k} |x_i - y_i|$
- $\Delta_k = \{ x \in \mathbb{R}^k \mid \sum x_i = 1 \}$ (A simplex in $\mathbb{R}^k$)
- For $e = (u,v)$ $\quad \sum_{i=1}^{k} z_e^i = \sum_{i=1}^{k} |x_u^i - x_v^i| = \|x_u - x_v\|_1$

$$\boxed{\begin{aligned} &\min \tfrac{1}{2} \sum_{e=(u,v)} c_e \|x_u - x_v\|_1 \\ &x_{s_i} = e_i \quad \forall s_i \in S \\ &x_v \in \Delta_k \quad \forall v \in V \end{aligned}}$$



A solution assigns a vector $x_v \in \Delta_k$ to vertex v

- Idea: take an LP sol^n $\{x_v\}$ and assign v to C_i if x_v is close to $e_i = x_{s_i}$ $\longrightarrow$ $x_v \in B(e_i, r)$

Ball of radius r

Note $B(e_i, 2) = V$

- Algo:
  - $C_i \leftarrow \emptyset \ \forall i \in [k]$
  - $X \leftarrow \emptyset$
  - pick random $r \in (0,1)$
  - pick random permutation π of $\{1 .. k\}$
  - for $i = 1 \cdots k$:

    $C_{\pi(i)} \leftarrow B(e_{\pi(i)}, r) - X$ ($C_{\pi(k)}$ actually has what is left)

    $X \leftarrow X \cup C_{\pi(i)}$

  - $F = \bigcup_{i=1}^{k} \delta(C_i)$